Journal of Algebra 331 (2011) 152–168

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Residually nilpotent groups whose closed subgroups are subnormal

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ARTICLE INFO

Article history:
Received 27 May 2010
Available online 9 December 2010
Communicated by E.I. Khukhro

MSC:
20E15
20F19

Keywords:
Closed subgroups
Subnormal subgroups
Subnormal intersection property
Residually nilpotent groups

ABSTRACT

We prove that a periodic residually nilpotent group G all of whose closed subgroups are subnormal is nilpotent. As a particular case, it follows that a periodic residually nilpotent group satisfying the subnormal intersection property is nilpotent.

1. Introduction

This paper grew from an attempt of extending a recent result of H. Smith [1] stating that *a periodic residually nilpotent group in which all subgroups are subnormal is nilpotent* (see also [2] for an alternative proof). To properly state our main result, let us introduce some definitions, which we will thoroughly use in the paper.

Definition 1. A family $\mathcal{N}$ of normal subgroups of a group G is called a *residual nilpotency system* for G if it satisfies the following conditions:

(P1) G/N is a nilpotent group for every $N \in \mathcal{N}$;
(P2) For all $N_1, N_2 \in \mathcal{N}$ there exists $N \in \mathcal{N}$ such that $N \leqslant N_1 \cap N_2$;
(P3) $\bigcap_{N\in\mathcal{N}} N = 1$.

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[1] The authors were supported by MIUR – PRIN project "Teoria dei gruppi e applicazioni".

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.10.022

Thus, a group G admits a residual nilpotency system if and only if it is residually nilpotent. We shall denote with $(G, \mathcal{N})$ the pair defined by a residually nilpotent group G, equipped with a given residual nilpotency system $\mathcal{N}$, and call it a residually nilpotent pair (r.n.p. for short).

Definition 2. Let $(G, \mathcal{N})$ be a residually nilpotent pair, and let H be a subgroup of G. The *closure* of H in $(G, \mathcal{N})$ is defined as the subgroup

$$cl_G(H) = \bigcap_{N \in \mathcal{N}} HN.$$

We then say that the subgroup H is *closed* if $H = cl_G(H)$.

Our main result is the following:

Theorem 1. *Let G be a periodic group admitting a residual nilpotency system such that all closed subgroups are subnormal. Then G is nilpotent.*

We remark that, in [3], H. Smith has constructed (non-periodic) residually nilpotent groups in which every subgroup is subnormal that are not nilpotent. Thus, the above result does not hold for non-periodic groups. However, we believe that much can be said in the general case as well, but leave this for further investigation.

The subnormal intersection property. A group G is said to satisfy the *Subnormal Intersection Property* (abbreviated with *s.i.p.*) if given any family $(H_\lambda)_{\lambda \in \Lambda}$ of subnormal subgroups of G, the intersection $\bigcap_{\lambda \in \Lambda} H_\lambda$ is again a subnormal subgroup of G. Following [4], we denote by $\mathfrak{S}_\infty$ the class of all groups satisfying the *s.i.p.* Clearly, all finite groups belong to $\mathfrak{S}_\infty$, and so the *s.i.p.* is a finiteness condition in the usual sense. In fact, it is not difficult to see that all groups with the minimal condition on subnormal subgroups $(\min - sn)$ satisfy *s.i.p.*; on the other hand, for instance, the infinite dihedral group does not satisfy such a property. An immediate corollary of our result is the following.

Corollary 1. *A periodic residually nilpotent group with the subnormal intersection property is nilpotent.*

In fact, the initial motivation of our work came as part of the project of beginning a more detailed investigation of locally nilpotent groups with the s.i.p. (the connection with local nilpotency is to be found in the rather easy observation – Lemma 9 – that the groups that comprise the object of our study are indeed Baer groups). As a consequence, we devote the last section of this paper, mainly composed by examples, to reviewing Corollary 1 in the framework of the theory of locally nilpotent groups and of groups with the s.i.p.. The interested reader may thus jump directly to that for more comments in this direction.

Topological groups. Clearly, a family $\mathcal{N}$ of normal subgroups of G as in Definition 1, determines an inverse system of nilpotent groups and a topology on G, which, by condition (P3), is Hausdorff: the closed subgroups in Definition 2 are just those subgroups of G that are closed in this topology. However, besides such small use of terminology, we have not taken a topological approach in this paper.

The notation is mostly standard. If H is a subnormal subgroup of G (written $H \lhd\lhd G$), we let $d(H, G)$ denote the *defect* of H in G, i.e. the shortest length of a series $H = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_d = G$. Also, we denote by $G^{(n)}$, $\zeta_n(G)$ and $\gamma_n(G)$, the n-th term of, respectively, the derived series, the upper and lower central series of G (and write $Z(G) = \zeta_1(G)$ for the center of G).

2. Closed subnormality

Let $(G, \mathcal{N})$ be a residually nilpotent pair, according with the definition given in the introduction: for a subgroup H of G we write $H \leqslant_c G$ if H is closed in $(G, \mathcal{N})$, and $H \trianglelefteqslant_c G$ if furthermore H is normal in G.

In this section we collect a series of basic and mostly straigthforward facts, that we will often use without any further reference. They are stated for groups with a residual nilpotent system, but their validity is in general much wider (for instance most of them do not depend on the assumption that the factors G/N – for $N \in \mathcal{N}$ – are nilpotent).

Lemma 2. *Let $(G, \mathcal{N})$ be a residually nilpotent pair. Then*

(i) *if $\{H_\lambda\}_{\lambda \in \Lambda}$ is a collection of closed subgroups, then $\bigcap_{\lambda \in \Lambda} H_\lambda$ is a closed subgroup;*
(ii) *if $H \leqslant K \leqslant G$, then $cl_G(H) \leqslant cl_G(K)$;*
(iii) *if $H \trianglelefteqslant K \leqslant G$, then $cl_G(H) \trianglelefteqslant cl_G(K)$.*

Lemma 3. *Let $\mathcal{N}$ be a family of normal subgroups of the group G satisfying* (P2) *and* (P3)*, and let $H \leqslant G$.*

(i) *If $Z = C_G(H)$, then $Z = \bigcap_{N \in \mathcal{N}} ZN$.*
(ii) *If H is finite, then $H = \bigcap_{N \in \mathcal{N}} HN$.*

Proof.

(i) Let $g \in \bigcap_{N \in \mathcal{N}} ZN$: then, for all $a \in H$, and all $N \in \mathcal{N}$,

$$[a, g] \in [H, ZN] \leqslant [H, N] \leqslant N.$$

Thus $[a, g] = 1$ and so $g \in C_G(H) = Z$.
(ii) Set $H_o = \bigcap_{N \in \mathcal{N}} HN$. For each $N \in \mathcal{N}$, $(H_o \cap N)H = H_o \cap NH = H_o$, and thus $H_o/(H_o \cap N) \simeq H/H \cap N$. Now, the intersection of all $H_o \cap N$, with $N \in \mathcal{N}$, is trivial, and since H is finite, it follows that H_o is finite. Thus, there exists a $K \in \mathcal{N}$, such that $H_o \cap K = 1$. This shows that $H_o = H$. □

Corollary 4. *Let $(G, \mathcal{N})$ be a residually nilpotent pair, and $H \leqslant G$. Then*

(i) *$C_G(H)$ is a closed subgroup of G;*
(ii) *if H is finite, H is a closed subgroup of G.*

Lemma 5. *Let $(G, \mathcal{N})$ be a residually nilpotent pair, and $H \leqslant G$. Then*

(i) *if $H \leqslant G$ is nilpotent of class c, then $cl_G(H)$ is nilpotent of class c;*
(ii) *if $H \leqslant G$ has finite exponent e, then $cl_G(H)$ has the same exponent;*
(iii) *if $H \lhd\lhd G$, then $cl_G(H) \lhd\lhd G$ and $d(cl_G(H), G) \leqslant d(H, G)$.*

Proof.

(i) Since $\gamma_{c+1}(H) = 1$ and $(P3)$ holds, we have

$$\gamma_{c+1}\bigl(cl_G(H)\bigr) = \bigl[cl_G(H),\, {}_c cl_G(H)\bigr] = \Bigl[\bigcap_{N \in \mathcal{N}} HN,\, {}_c \bigcap_{N \in \mathcal{N}} HN\Bigr] \leqslant \bigcap_{n \in \mathcal{N}} [H,\, {}_c H]N = \bigcap_{N \in \mathcal{N}} N = 1.$$

(ii) Let $x \in cl_G(H)$. Then, for $N \in \mathcal{N}$, $x = h_n n$, where $h_n \in H$ and $n \in N$. Thus, $x^e = (h_n n)^e = h_n^e n' = n' \in N$. This applies to every $N \in \mathcal{N}$, therefore $x^e \in \bigcap_{N \in \mathcal{N}} N = 1$, and 2 follows.

(iii) Let $H \lhd\lhd G$, and $d = d(H, G)$. Then, for every $N \in \mathcal{N}$, HN/N is subnormal of defect at most d in G/N. This implies $[G,_d NH] \leqslant N[G,_d H] \leqslant NH$ for every $N \in \mathcal{N}$; thus $[G,_d cl_G(H)] \leqslant cl_G(H)$, which is what we wanted. □

Let $(G, \mathcal{N})$ be a residually nilpotent pair. Then every subgroup H of G naturally inherits a residual nilpotency system $\mathcal{N}_H = \{N \cap H\}_{N \in \mathcal{N}}$, and thus $(H, \mathcal{N}_H)$ is a residually nilpotent pair. When referring to a subgroup H of G, we will always tacitly assume that H is endowed with such induced system. Similarly, if $H \trianglelefteqslant_c G$, then the family $\mathcal{N}_{G/H} = \{HN/H\}_{N \in \mathcal{N}}$ is a residual nilpotency system for G/H (and we will always assume G/H endowed with such residual system).

Lemma 6. *Let $(G, \mathcal{N})$ be a residually nilpotent pair. Then,*

(i) *For every $K \leqslant H \leqslant G$, $cl_H(K) = H \cap cl_G(K)$;*
(ii) *If $K \leqslant H \leqslant G$ and $K \leqslant_c G$, then $K \leqslant_c H$;*
(iii) *If $K \leqslant_c H \leqslant_c G$, then $K \leqslant_c G$.*

Proof. Let $K \leqslant H \leqslant G$. Then, by Lemma 2,

$$cl_H(K) = \bigcap_{N \in \mathcal{N}} K(N \cap H) = \left(\bigcap_{N \in \mathbb{N}} KN \right) \cap H = cl_G(K) \cap H,$$

thus proving point (i). Points (ii) and (iii) now follow immediately. □

Lemma 7. *Let $(G, \mathcal{N})$ be a residually nilpotent pair, and let $H \trianglelefteqslant_c G$. Then for every $H \leqslant K \leqslant G$,*

$$cl_{G/H}(K/H) = cl_G(K)/H.$$

Proof. Immediate from the definitions. □

From this and point (i) of Lemma 3 one easily deduce the following.

Lemma 8. *Let $(G, \mathcal{N})$ be a residually nilpotent pair, and let $H \trianglelefteqslant_c G$. Then for every $n \in \mathbb{N}$,*

(i) *$\zeta_n(H) \leqslant_c G$;*
(ii) *the subgroup Z_n defined by $Z_n/H = \zeta_n(G/H)$ is closed in G.*

We may now define the class of objects which is the main interest of this paper.

Definition 3. A residually nilpotent pair $(G, \mathcal{N})$ is *subnormal-closed* if every closed subgroup of it is subnormal in G. We denote by $\mathfrak{S}_c$ the class of all groups G that admit a residual nilpotency system $\mathcal{N}$, such that $(G, \mathcal{N})$ is subnormal-closed.

One observes at once that if the residually nilpotent pair $(G, \mathcal{N})$ is subnormal-closed, then for every subgroup H, the pair $(H, \mathcal{N}_H)$ is subnormal-closed: in fact, by Lemma 6, for any closed subgroup K of H,

$$K = cl_H(K) = cl_G(K) \cap H \lhd\lhd H.$$

By Lemma 7, a similar remark applies for every factor pair $(G/N, \mathcal{N}_{G/N})$, with $N \trianglelefteqslant_c G$.

Thus, in particular, we deduce that the class $\mathfrak{S}_c$ is subgroup closed (quotient closure is not obvious at this stage).

The starting point in studying $\mathfrak{S}_c$-groups is the elementary but fundamental observation that they are locally nilpotent. In fact (and in a strong sense), they even are Baer groups: that is, groups in which every cyclic subgroup is subnormal.

Lemma 9. *Let G be a group in the class $\mathfrak{S}_c$. Then every nilpotent subgroup of G is subnormal. In particular, G is a Baer group.*

Proof. Let $G \in \mathfrak{S}_c$, and let $\mathcal{N}$ be a residual nilpotency system in G, such that every closed subgroup in $(G, \mathcal{N})$ is subnormal. Let H be a nilpotent subgroup of G. Then $cl_G(H)$ is subnormal in G, and moreover, by Lemma 5, it is nilpotent. Thus, $H \lhd\lhd cl_G(H) \lhd\lhd G$, and so $H \lhd\lhd G$. $\square$

3. Preliminaries

Our proof follows the lines of that of Smith's Theorem as given in [2]: in particular, we need preliminary information on p-groups of finite exponent (Section 4), that we will obtain by using Möhres's ideas for the corresponding case in the class of groups with all subgroups subnormal (see [5] and [6]). Let us begin by recalling an easy fact.

Lemma 10. *Let G be a group, $H \leqslant G$, and let V be a finitely generated subgroup of H. If all finitely generated subgroups of H that contain V are subnormal in G with defect bounded by an integer d, then H is subnormal in G of defect at most d.*

The following lemma is an easy rephrasing of an argument originally due to C. Brookes, which has become a standard trick in many articles on groups with many subnormal subgroups. For the convenience of the reader, we include a proof.

Lemma 11. *Assume that the residually nilpotent pair $(G, \mathcal{N})$ is subnormal-closed, and let Θ be a family of closed subgroups such that $G \in \Theta$. Then there exist a $H \in \Theta$, a finitely generated subgroup F of H, and a positive integer d, such that every $K \in \Theta$ with $F \leqslant K \leqslant H$ has defect at most d in H.*

Proof. Let $(G, \mathcal{N})$ be a counterexample. By an inductive procedure we construct two chains of subgroups

$$\{1\} = F_0 \leqslant F_1 \leqslant \cdots \leqslant F_i \leqslant F_{i+1} \leqslant \cdots$$

$$G = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_i \geqslant H_{i+1} \geqslant \cdots$$

such that, for each $i, j \in \mathbb{N}$, F_i is finitely generated, $F_i \leqslant H_j \in \Theta$ and $[H_i, {}_i F_{i+1}] \nleqslant H_{i+1}$.

Set $F_0 = \{1\}$, $H_0 = G$, and suppose we have already defined $F_0, \ldots, F_i$ and $H_0, \ldots, H_i$. Since $F_i \leqslant H_i \in \Theta$, $H_i \lhd\lhd G$, and G is a counterexample, there exists a subgroup $H_{i+1} \in \Theta$, with $F_i \leqslant H_{i+1} \leqslant H_i$, and $d(H_{i+1}, H_i) \geqslant i+1$. Then $[H_i, {}_i Hi+1]$ is not contained in H_{i+1} and so there exists a finitely generated subgroup K of H_{i+1} such that $[H_i, {}_i K] \nleqslant H_{i+1}$. We put $F_{i+1} = \langle F_i, K \rangle$. Then F_{i+1} is finitely generated, $F_i \leqslant F_{i+1} \leqslant H_{i+1}$, and $[H_i, {}_i F_{i+1}] \nleqslant H_{i+1}$.

By induction, we thus construct the two chains $\{F_i\}_{i \in \mathbb{N}}$, $\{H_i\}_{i \in \mathbb{N}}$ with the desired properties. We then put

$$H = \bigcap_{i \in \mathbb{N}} H_i.$$

Then $\bigcup_{i\in\mathbb{N}} F_i \leqslant H$, and H is closed in $(G,\mathcal{N})$. Hence, as $(G,\mathcal{N})$ is subnormal-closed, H is subnormal in G, and so there exists an integer k such that $[G,_k H] \leqslant H$. In particular we have

$$[G,\ _kF_{k+1}] \leqslant [G,\ _kH] \leqslant H \leqslant H_{k+1}$$

which contradicts the choice of F_{k+1}. $\square$

Another fundamental tool in this kind of investigation is a famous result of Roseblade [7].

Theorem 2 *(Roseblade). Let G be a group. Suppose that there exists a positive integer d such that all subgroups of G are subnormal of defect at most d in G. Then G is nilpotent and its nilpotency class is bounded by a function of d.*

This statement may clearly be adapted to our context, in the following form:

A group G admitting a residual nilpotency system such that all closed subgroups are subnormal of defect at most d, is nilpotent of nilpotency class bounded by a function of d.

Indeed, when we refer to Roseblade's Theorem, it will always be meant in the sense of the last remark. We will also need an extension due to E. Detomi [8].

Proposition 12 *(Detomi). Let G be a periodic locally nilpotent group. Assume that there exist a finite subgroup F of G and $d \in \mathbb{N}$, such that every subgroup of G containing F is subnormal of defect at most d in G. Then $\gamma_{\beta(d)}(G)$ is finite for a positive integer $\beta(d)$ depending only on d. In particular, G is nilpotent and its nilpotency class is bounded by a function that depends only on d and $|F|$.*

Remark 1. Now, let G be a periodic group in $\mathfrak{S}_c$. Then G is locally nilpotent by Lemma 9, and so it is the direct product of its primary components G_p (for $p \in \mathbb{P}$, the set of all prime numbers). It follows that, for any residual nilpotency system $\mathcal{N}$, G_p is closed in $(G,\mathcal{N})$. In fact, given $\mathcal{N}$, let $g \in cl_G(G_p)$, and suppose that $m = |g|$ is coprime to p. For any $N \in \mathcal{N}$, $g = hx$ with $h \in G_p$, $x \in N$: so $1 = g^m = h^m n'$ (with $n' \in N$); forcing, as h is a p-element, $h \in N$. Hence $g \in \bigcap_{N\in\mathcal{N}} N = 1$, showing that g must be a p-element and thus belong to G_p.

This fact will be applied without any further reference. In particular it is instrumental in reducing to primary groups, which we do in the next lemma.

Lemma 13. *Let G be a periodic group in $\mathfrak{S}_c$; then there exists a positive integer n, such that all but finitely many primary components of G are nilpotent of nilpotency class at most n.*

Proof. Let G be as in the hypothesis; then G is the direct product of its primary components G_p, and, as observed above, every component G_p is closed with respect to any residual nilpotency system $\mathcal{N}$ of G.

Suppose, by contradiction, that, for every $n \in \mathbb{N}$ there are infinitely many components G_p, such that $\gamma_n(G_p) \neq 1$. Since $G_p \leqslant_c G$, by Roseblade's Theorem 2, we can find distinct primes p_n, and for each $n \in \mathbb{N}$, a closed subgroup T_n of G_{p_n} ($T_n \leqslant_c G$ by Lemma 6) of defect at least n in G_{p_n} (observe that this is possible because G is a Baer group). For each n, we set $H_n = (Dir_{q\neq p_n} G_q) \times T_n$; then H_n is subnormal in G of defect at least n and it is easily seen to be closed.

By Lemma 2, the subgroup

$$H = \bigcap_{n\in\mathbb{N}} H_n$$

is closed in G and so, since $G \in \mathfrak{S}_c$, it is subnormal of defect, say, d. But then, for $m \geqslant d+1$,

$$T_m = H \cap G_{p_m} \geqslant [G, {}_d H] \cap G_{p_m} \geqslant [G_{p_m}, {}_d T_m] \geqslant [G_{p_m}, {}_m T_m],$$

a contradiction. $\square$

However, our next result does not require the group to be periodic.

Theorem 3. *Every group in $\mathfrak{S}_c$ is soluble.*

Proof. Let $G \in \mathfrak{S}_c$, and $\mathcal{N}$ a residual nilpotency system in G such that $(G, \mathcal{N})$ is subnormal-closed. Suppose by contradiction that G is not soluble. By Lemma 11 applied to the family Θ of all closed non-soluble subgroups of $(G, \mathcal{N})$, there exist $H \in \Theta$, a finitely generated subgroup F of H, and a positive integer d, such that all non-soluble closed subgroups of H containing F have defect at most d in H. By Lemma 6, we may well replace G by H, and assume that d is minimal for a counterexample.

Now, observe that every element $N \in \mathcal{N}$ is not soluble. Hence, for $N \in \mathcal{N}$, and every subgroup $Y \geqslant F$, YN is a closed non-soluble subgroup of H, that is $YN \in \Theta$. Therefore, for every $N \in \mathcal{N}$, and every $Y \geqslant F$, YN is subnormal of defect at most d in H.

We have $d \neq 1$. In fact, if $d = 1$, then, by what we have just observed, for every $N \in \mathcal{N}$, $FN \trianglelefteqslant G$, all subgroups of H/N containing FN/N are normal. That is H/FN is a Dedekind group, and therefore

$$H^{(2)} \leqslant \bigcap_{N \in \mathcal{N}} FN = cl_H(F).$$

Now, since H is a Baer group, F is nilpotent and so, by Lemma 5, $cl_H(F)$ is nilpotent, and thus soluble of derived length, say, t. We then have

$$H^{(2+t)} \leqslant \bigcap_{N \in \mathcal{N}} \big(cl_H(F)\big)^{(t)} = 1,$$

contradicting the choice of H.

Let now $d \geqslant 1$ and let $F \leqslant K \leqslant_c cl_H(F^H)$, with K not soluble. Since $K \leqslant_c G$ by Lemma 6, K is subnormal of defect at most d in H. Thus, for every $N \in \mathcal{N}$,

$$\big[F^H N, {}_{d-1} K\big] \leqslant \big[F^H, {}_{d-1} K\big] N \leqslant \big[K^H, {}_{d-1} K\big] N \leqslant KN,$$

yielding

$$\big[cl_H\big(F^H\big), {}_{d-1} K\big] \leqslant \bigcap_{N \in \mathcal{N}} \big[F^H N, {}_{d-1} K\big] \leqslant \bigcap_{N \in \mathcal{N}} KN = K.$$

Hence, K is subnormal of defect at most $d-1$ in $cl_H(F^H)$. By the minimality of d, we deduce that $cl_H(F^H)$ is soluble. Now, $cl_H(F^H) \trianglelefteqslant H$ by Lemma 5, and all closed subgroups of $H/cl_H(F^H)$ are subnormal of defect at most d. By Rosebòlade's Theorem, there is an integer k such that $H^{(k)} \leqslant cl_H(F^H)$. Then, if t is the derived length of $cl_H(F^H)$,

$$H^{(k+t)} \leqslant \big(cl_H\big(F^H\big)\big)^{(t)} = 1,$$

a contradiction that concludes the proof. $\square$

We now treat the special case of residually finite groups.

Definition 4. A residually nilpotent pair $(G, \mathcal{N})$ is said to be *residually finite* if the intersection of all closed subgroup of finite index is the identity element.

In the next proof we will use the standard fact that a Baer group with a nilpotent subgroup of finite index is itself nilpotent.

Proposition 14. *Let the residually nilpotent pair $(G, \mathcal{N})$ be subnormal-closed and residually finite. If G is periodic, then G is nilpotent.*

Proof. Let $(G, \mathcal{N})$ be as in the hypothesis, and suppose G periodic. We apply Lemma 11 to the family Θ of all closed subgroups of G of finite index. Then there exist $H \in \Theta$, a finitely generated (and thus finite) subgroup F of H, and a positive integer d, such that every $F \leqslant K \leqslant H$, with $K \in \Theta$, has defect at most d in H.

Let V be a finitely generated (and hence finite) subgroup of H containing F. Since G is a Baer group, V is subnormal in H. We show that V has defect at most d in H. Let N be a closed normal subgroup of H of finite index; then VN/N is finite and $F \leqslant VN \leqslant_c H$, so $VN \in \Theta$. Thus, VN is subnormal of defect at most d in H. Consequently, denoting by $\mathcal{F}$ the set of all closed normal subgroups of H of finite index, the subgroup

$$V_o = \bigcap_{N \in \mathcal{F}} VN$$

is subnormal of defect at most d in H.

Now, by assumption, $\bigcap_{N \in \mathcal{F}} N = 1$, and thus it follows from Lemma 3 that $V_0 = V$, proving that V has defect at most d in H.

By Lemma 10 we thus deduce that all subgroups of H containing F are subnormal of defect at most d, and by Proposition 12, we conclude that H is nilpotent. Since H has finite index in the Baer group G, by the reminder above we get that G is nilpotent. $\square$

Of course, Proposition 14 will be superseded by Theorem 1. We note that, also in this simpler case, we cannot remove the assumption that G is periodic. In fact, as proved by H. Smith [3], there even exist residually finite non-periodic groups that are not nilpotent and have all subgroups subnormal.

4. $\mathfrak{S}_c$-groups of finite exponent

In this section, we consider soluble p-groups of finite exponent. By a well known fact (see [4, Theorem 7.17]), such groups are Baer groups. The first and main case to be studied is that of an extension of an elementary abelian p-group by another elementary abelian p-group. For a fixed prime p, following Möhres [5], we set the following notation: $(G, A) \in \Phi$ means that A is a normal subgroup of the group G, and both A and G/A are elementary abelian p-groups.

We rephrase some extremely useful lemmas from the work of Möhres.

Lemma 15. *(Cf. [5, 1.4, 1.5].) Let $(G, A) \in \Phi$, and let $n \in \mathbb{N}$.*

(i) *If $[A, {}_{n(p-1)}G] \neq 1$, then there exist $x_1, x_2, \dots, x_n \in G$ such that $[A, {}_{p-1}x_1, \dots, {}_{p-1}x_n] \neq 1$;*
(ii) *Let $x_1, x_2, \dots, x_n \in G$ be such that $[A, {}_{p-1}x_1, \dots, {}_{p-1}x_n] \neq 1$; and let $y_1, \dots, y_m \in \langle A, x_1, x_2, \dots, x_n\rangle$ with $Ay_1, \dots, Ay_m$ linearly independent in G/A. Then $[A, {}_{p-1}y_1, \dots, {}_{p-1}y_m] \neq 1$.*

Before proceeding, we recall the easy fact that a nilpotent by finite p-group of finite exponent is nilpotent.

Lemma 16. *Let $(G, A) \in \Phi$, $X \leqslant G$ with $A \leqslant X$ and X/A finite, and let $f(n)$ be a positive integer valued function on $\mathbb{N}$. Assume that G is not nilpotent. Then there exists a chain of subgroups $X = X_0 < X_1 < X_2 < \cdots < X_n < \cdots$ such that, for all $i \in \mathbb{N}$,*

(1) *X_i/A is finite, and $|X_{i+1}/X_i| = p^{f(i)}$;*
(2) *for all $H \leqslant G$, if for some $i \in \mathbb{N}$, and some $1 \leqslant k \leqslant f(i)$, we have $|X_i(AH \cap X_{i+1})/X_i| \geqslant p^k$, then $[A,\, _{k(p-1)}H] \neq 1$.*

Proof. Set $X_0 = X$, and suppose that, for some $n \in \mathbb{N}$, we have already found subgroups $X_0, X_1, \ldots, X_n$ with the desired properties. Let Y/A be a complement of X_n/A in G/A. Since G is not nilpotent and $|G : Y| < \infty$, Y is not nilpotent. On the other hand, since X_n/A is finite, X_n is nilpotent, and there exists $c \in \mathbb{N}$ such that $[A,\, _cX_n] = 1$. Let $C = C_A(X_n)$, and suppose that $[C,\, _sY] = 1$ for some s. Then, since A and G/A are abelian,

$$[A,\, _sY,\, _{c-1}X_n] = [A,\, _{c-1}X_n,\, _sY] \leqslant [C,\, _sY] = 1;$$

and so, by an easy induction, $[A,\, _{cs}Y] = 1$, a contradiction.

Hence, in particular, $C \nleqslant \zeta_{f(n)(p-1)}(Y)$. Then, by Lemma 15, there exist $a_{n+1} \in C$, and $y_1, y_2, \ldots, y_{f(n)} \in Y$ such that

$$[a_{n+1},\, _{p-1}y_1, \ldots,\, _{p-1}y_{f(n)}] \neq 1$$

and $y_1A, \ldots, y_nA$ are independent. We set $X_{n+1} = \langle X_n, y_1, \ldots, y_{f(n)} \rangle$, and show that it satisfies the required properties.

Let $H \leqslant X_{n+1}$ such that X_nH/X_n has order p^k with $1 \leqslant k \leqslant f(n)$. Then there exist k linearly independent elements $z_1A, \ldots, z_kA$ of G/A, with $z_1, \ldots, z_k \in H$, such that

$$\frac{HX_n}{A} = \frac{X_n}{A} \times \langle z_1A, \ldots, z_kA \rangle.$$

By the construction of X_{n+1}, for each $i = 1, \ldots, k$, $z_i = b_iu_i$, with $b_i \in X_n$, and $u_i \in \langle y_1, \ldots, y_{f(n)} \rangle$ (using the notation of the first part of the proof). Finally, since $[a_{n+1}, X_n] = 1$, we get

$$[a_{n+1},\, _{p-1}z_1, \ldots,\, _{p-1}z_k] = [a_{n+1},\, _{p-1}u_1, \ldots,\, _{p-1}u_k] \neq 1$$

by choice of $y_1, \ldots, y_{f(n)}$ and part 2) of Lemma 15. Therefore $[A,\, _{k(p-1)}H] \neq 1$, and this completes the proof. □

Another rather elementary fact that we need is the following.

Lemma 17. *Let $(G, A) \in \Phi$, and $x_1, \ldots, x_n \in G$. Then $|\langle x_1, \ldots, x_n \rangle| \leqslant p^{\omega(n)}$; where $\omega(1) = 2$, and, for $n \geqslant 1$, $\omega(n) = 2n + p^{2n}\binom{n}{2}$.*

Proof. See e.g. (cf. [6, Lemma 1.1]). □

The next lemma also essentially comes from Möhres.

Lemma 18. *For every $n, m \geqslant 1$, there exists $\psi(n, m) \in \mathbb{N}$, such that the following holds.*

If $(G, A) \in \Phi$ with $|G/A| \geqslant \psi(n, m)$, $V = \langle x_1, \ldots, x_n \rangle \leqslant G$ and $a \in A \setminus V$; then there exist elements $y_1, \ldots, y_m \in G$, such that $a \notin U = \langle V, y_1, \ldots, y_m \rangle$, and $|AU : AV| = p^m$.

Proof. By a result of Möhres [6, Satz 2.2] there exists a function $\mu : \mathbb{N} \to \mathbb{N}$ such that if $(G, A) \in \Phi$, $|G/A| \geqslant \mu(n)$, and $V \leqslant G$ with $|V| \leqslant p^n$, then

$$\bigcap_{y \in G \setminus AU} \langle V, y \rangle = V.$$

For $n, m \geqslant 1$, we set $\psi(n,m) = \mu(\omega(n+m-1))$ (where ω is the function defined in Lemma 17), and show that it satisfies the required property by induction on m. Let $x_1, \dots, x_n \in G$, $V = \langle x_1, \dots, x_n\rangle$, and $a \in A \setminus V$. If $m = 1$, $\psi(n,1) = \mu(\omega(n))$ and the claim follows from Möhres's Theorem. Let $m \geqslant 2$, then, as $\psi(n,m) \geqslant \psi(n,m-1)$, by inductive assumption there exist $y_1, \dots, y_{m-1}$ such that $a \notin T = \langle V, y_1, \dots, y_{m-1}\rangle$ and $|AT : AV| = p^{m-1}$. Now, by Lemma 17, $|T| \leqslant p^{\omega(n+m-1)}$, and so the result of Möhres again implies the existence of $y_m \in G$ with $a \notin U = \langle T, y_m\rangle = \langle V, y_1, \dots, y_{m-1}, y_m\rangle$ and $|AU : AT| = p$. Thus $|AU : AV| = |AU : AT||AT : AV| = p^m$, and the lemma is proved. □

One further preparatory result that we need is of different nature, following easily from P. Hall's nilpotency criterion (see [4, Theorem 2.27]).

Proposition 19 *(P. Hall). Let H be a normal subgroup of the group G. If both H and G/H' are nilpotent, of nilpotency class c and d respectively, then G is nilpotent of nilpotency class at most $\binom{c+1}{2}d - \binom{c}{2}$.*

Lemma 20. *Let $(G, \mathcal{N})$ be a residually nilpotent pair, with G a p-group, and let H be a normal subgroup of G of finite exponent p^e. Suppose that H is nilpotent of class c and $G/cl_G(H'H^p)$ is nilpotent of class d. Then, G is nilpotent of class bounded by a function $\eta(e, c, d)$.*

Proof. Write $M = H'H^p$, and $R = cl_G(M)$. Let $N \in \mathcal{N}$, and $\overline{H} = HN/N$. Then, clearly, $\overline{H}$ has exponent dividing p^e, is nilpotent of class at most c, and $\overline{H}'\overline{H}^p = MN/N = RN/N$, whence $\overline{G}/\overline{H}'\overline{H}^p$ is nilpotent of class at most d. As $\bigcap_{N\in\mathcal{N}} N = 1$, we may well just suppose $N = 1$. Then $M = H'H^p$ is closed, and $\gamma_{d+1}(G) \leqslant M$; hence, by standard commutator calculus,

$$\gamma_{2d+1}(G) \leqslant \left[H'H^p,\, {}_dG\right] \leqslant H'[H,\, {}_dG]^p \leqslant H'H^{p^2}.$$

Thus, since H has exponent p^e, $\gamma_{ed+1}(G) \leqslant H'$, and we conclude by appealing to P. Hall's criterion. □

We are finally ready to deal with the case $(G, A) \in \Phi$.

Lemma 21. *Let $(G, A) \in \Phi$. Suppose that G belongs to $\mathfrak{S}_c$; then G is nilpotent.*

Proof. Let $\mathcal{N}$ be a residual nilpotency system in G such that $(G, \mathcal{N})$ is subnormal-closed.

Suppose, by contradiction, that G is not nilpotent. Then, by Lemma 11 (possibly replacing G by a suitable non-nilpotent closed subgroup), we may assume that there exist a positive integer d and a finite subgroup F of G, such that all non-nilpotent closed subgroups of G containing F have defect at most d in G.

We prove that all subgroups of G containing F are subnormal of defect at most d. Suppose the contrary; then, by Lemma 10, there exists a finitely generated subgroup V containing F, whose defect is larger than d (V is certainly subnormal because G is a Baer group).

Observe that, if $N \in \mathcal{N}$ is not nilpotent, then VN is a non-nilpotent closed subgroup containing F, and so $d(VN, G) \leqslant d$. Since, by Lemma 3, $V = \bigcap_{N\in\mathcal{N}} VN$, we deduce that

(1) there exists $N \in \mathcal{N}$ such that N is nilpotent and VN has defect larger than d in G.

Let such $N \in \mathcal{N}$ be fixed, and let $H = AN$. Then, H is a nilpotent closed subgroup of G of exponent at most p^2. Hence, setting $R = cl_G(H'H^p)$, by choice of G as a counterexample and Lemma 20, we have that G/R is not nilpotent. Observe also that $R \leqslant N$. Now, clearly, $(G/R, \mathcal{N}_{G/R})$ is subnormal-closed. Moreover, $(G/R, H/R) \in \Phi$, and, since $VN \geqslant R$, $d(VN/R, G/R) > d$. Thus, we may replace (G, A) with $(G/R, H/R)$, F with FR/R, N with N/R, and thus assume

(2) A is closed and $N \leqslant A$.

Now, since VN has defect greater than d in G,

(3) there exists an element $a \in A$ such that $a \in [G, {}_d VN] \setminus VN$.

Let V be generated by k elements of G, and define a function $f(i)$ (for $i \in \mathbb{N}$) by setting:

$$f(i) = \psi\left(k + \frac{i(i+1)}{2}, i+1\right),$$

where ψ is defined in Lemma 18. Now, as G is not nilpotent, there exists a chain of subgroups

$$AV = X_0 < X_1 < X_2 < \cdots < X_n < \cdots$$

satisfying the properties in the statement of Lemma 16, with respect to the above defined function $f(i)$.

Let bars denote the image of elements and subgroups of G modulo N. Then, by the choice of the function $f(i)$ and Lemma 18, there exists a chain of subgroups

$$\overline{V_0} < \overline{V_1} < \overline{V_2} < \cdots < \overline{V_i} < \cdots$$

of $\overline{G} = G/N$, with $V_0 = V$, such that, for every $i \in \mathbb{N}$, V_i is finitely generated (and thus finite), $|V_i X_i / X_i| = p^i$, and $\bar{a} \notin \overline{V_i}$.

In particular, a does not belong to the subgroup $W = \bigcup_{i \in \mathbb{N}} NV_i$. Now, since $N \leqslant W$, W is closed in $(G, \mathcal{N})$, and thus subnormal in G. Now,

$$a \in [G, {}_d VN] \setminus W \leqslant [G, {}_d W] \setminus W,$$

and so W has defect $s > d$ in G. It follows that, since it contains $V \geqslant F$, W is nilpotent. Let c be the nilpotency class of W; then

$$[A, {}_{s+c} W] = [A, {}_s W, {}_c W] \leqslant [W, {}_c W] = 1.$$

On the other hand, by Lemma 16, for all $i \in \mathbb{N}$ we have

$$[A, {}_{i(p-1)} W] \geqslant [A, {}_{i(p-1)} V_i] \neq 1.$$

By this contradiction all subgroups containing F are subnormal of defect at most d. Since F is finite and G is periodic and locally nilpotent, we conclude by Proposition 12 that G is nilpotent. $\square$

In order to extend Lemma 21 to arbitrary finite exponent, we notice the following fact, which is an easy application of Lemma 5.

Lemma 22. *Let p be a prime, and let $(G, \mathcal{N})$ be a residually nilpotent pair. If there is a normal series*

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that G_i/G_{i-1} is p-elementary abelian for each $i = 1, \ldots, n$, then there is such a series (of the same length) consisting of closed subgroups.

Here is the principal result of this section.

Proposition 23. *Every $\mathfrak{S}_c$-group of finite exponent is nilpotent.*

Proof. Let G a group of finite exponent belonging to $\mathfrak{S}_c$, and let $\mathcal{N}$ be a residual nilpotency system in G such that $(G, \mathcal{N})$ is subnormal-closed. By Lemma 9, G is a Baer group and so, having finite exponent, it is the direct product of a finite number of primary components, each of which is closed (see Lemma 1). By Theorem 3, G is soluble, and thus it admits a finite normal series whose factors are elementary abelian p-groups, and whose terms are closed in G (by Lemma 22). We proceed by induction on the length t of a shortest such series in G.

If $t = 1$ then G is elementary abelian, and there is nothing to prove. Thus, suppose $t > 1$, and let H be a normal closed subgroup of G such that G/H is elementary abelian, and H has a normal series of the above kind of length $t - 1$. By inductive assumption, H is nilpotent. Let $R = cl_G(H'H^p)$; then $R \trianglelefteq_c G$, G/R belongs to $\mathfrak{S}_c$ (via the system $\mathcal{N}_{G/R}$), and $(G/R, H/R) \in \Phi$. By Lemma 21, G/R is nilpotent. By Lemma 20, we obtain the theorem. □

5. The main result

Let us start this section by recalling some other well-known facts.

Lemma 24. *Let N be a normal abelian subgroup of the group G, and let $x \in G$. Suppose that there exist $1 \leqslant m, n \in \mathbb{N}$ such that $[N, x^m] = 1 = [N, {}_n x]$. Then x centralizes $N^{m^{n-1}}$.*

Proof. We argue by induction on n. If $n = 1$ we have nothing to prove. Thus, let $n \geqslant 2$, and set $B = [N, x]$. Then $[B, {}_{n-1} x] = 1$, whence, by inductive assumption, using the fact that N is abelian,

$$\left[N^{m^{n-2}}, x, x\right] = \left[[N, x]^{m^{n-2}}, x\right] = \left[B^{m^{n-2}}, x\right] = 1.$$

Now, let $b \in N^{m^{n-2}}$. Then, since $[b, x, x] = 1 = [b, x, b]$, by well known rule for commutators, we have $[b^m, x] = [b, x]^m = [b, x^m] = 1$. Therefore, $[N^{m^{n-1}}, x] = [(N^{m^{n-2}})^m, x] = 1$, as wanted. □

Corollary 25. *Let A be a normal abelian subgroup of the periodic Baer group G, and write $A^\omega = \bigcap_{n \in \mathbb{N}} A^n$. Then $A^\omega \leqslant Z(G)$.*

Proof. Since G is a Baer group, every cyclic subgroup of G is subnormal. Therefore, for all $x \in G$, there exists a positive integer $n = n(x)$ such that $[A, {}_n x] = 1$. By Lemma 24, x centralizes $A^{m^{n-1}}$, where $m = |x|$, and so x centralizes A^ω. □

We also need an easy application of P. Hall's criterion, suited to our purposes.

Lemma 26. *Let $(G, \mathcal{N})$ be a residually nilpotent pair, $H \trianglelefteq G$, and $R = cl_G(H')$. If both H and G/R are nilpotent, then G is nilpotent.*

Proof. Let c and d be, respectively, the nilpotency class of H and of G/R. Let $N \in \mathcal{N}$; then G/N has a normal subgroup HN/N of nilpotency class at most c. Now, $cl_{G/N}((HN/N)') = (HN/N)'$, and $\frac{G/N}{H'N/N}$ is nilpotent of class at most d. By Proposition 19, G/N is nilpotent of class at most δ, where $\delta = \binom{c+1}{2}d - \binom{c}{2}$ is independent on the choice of $N \in \mathcal{N}$. Hence

$$\gamma_{\delta+1}(G) \leqslant \bigcap_{N \in \mathcal{N}} N = 1$$

as wanted. □

The proof of our last auxiliary result follows closely that of Möhres for the case of $\mathfrak{N}_1$-groups [9, Satz 12], and thus we will not repeat all the details.

Lemma 27. *Let G be a periodic group in $\mathfrak{S}_c$. If G is the extension of a nilpotent group by a group of finite exponent, then G is nilpotent.*

Proof. Let G be a periodic $\mathfrak{S}_c$-group, with a normal nilpotent subgroup M such that G/M has finite exponent, and $\mathcal{N}$ a residual nilpotency system in G such that $(G,\mathcal{N})$ is subnormal-closed. By Lemmas 9 and 1 we may assume that G is a p-group for some prime p. Also, by Theorem 3, G is soluble, and G/M has a finite normal series with elementary abelian factors. By Lemma 5, we may assume that M is closed, and by Lemma 22 that the terms of such a series are also closed. Assume, by contradiction, that G is not nilpotent. Then, by arguing on the length of this series, we have that G/M is elementary abelian. Finally, by Lemma 26, we reduce to the case in which M is abelian.

We first show that G does not satisfy any Engel condition. In fact, suppose that there exists $n \in \mathbb{N}$ such that $[M, {}_n x] = 1$ for all $x \in G$. Since G/M has exponent p and M is abelian, $[M, x^p] = 1$ for all $x \in G$; hence, by Lemma 24, $M^{p^{n-1}} \leqslant Z(G)$. By Lemma 8, $G/Z(G)$ is a $\mathfrak{S}_c$-group of finite exponent, and so $G/Z(G)$ is nilpotent by Proposition 23. Thus G is nilpotent.

By using this fact, one shows that there exists a non-nilpotent closed subgroup K of G, with $M \leqslant K$, such that a subgroup S of K is nilpotent if and only if MS/M is finite. The proof of this goes exactly as in Lemma 5 of [9], and so we do not reproduce it here.

Then, one applies Lemma 11: since $(K,\mathcal{N}_K)$ is subnormal-closed and K is not nilpotent, there exists a non-nilpotent closed subgroup H of K, a finite subgroup F of H and a $d \in \mathbb{N}$, such that every non-nilpotent closed subgroup of H containing F has defect at most d in H. It is clear that we may well assume $H = K$. Let V be a finitely generated subgroup of K with $F \leqslant V$ and suppose that the defect of V in K is strictly larger than d. Then, there exists an element $a \in [K, {}_d V] \setminus V$. As in the proof of Theorem 21, there exists some $N \in \mathcal{N}$, with N nilpotent, such that $a \notin VN$. Since K/N is a nilpotent p-group, by Satz 6 in [9] there exists $S/N \leqslant K/N$ such that $V \leqslant S$, $a \notin S$, and $|SM/N : M/N| = |SM : M|$ is infinite. As $S \geqslant N$, S is closed and thus subnormal in K. On the other hand, by choice of K, S is not nilpotent, and thus, since $F \leqslant V \leqslant S$, S has defect at most d in K, forcing

$$a \in [K, {}_d V] \leqslant [K, {}_d S] \leqslant S$$

against the fact that $a \notin S$.

The last contradiction shows that every finitely generated subgroup of K containing F is subnormal of defect at most d. Then, by Lemma 10, all subgroups of K containing F are subnormal of defect at most d, and so, by Proposition 12, K is nilpotent, which is the final contradiction. □

Let H be a subgroup of the group G. We write $H \leqslant_b G$ if there exists an integer $m \geqslant 1$ such that $g^m \in H$ for all $g \in G$. This is equivalent to saying that G/H_G is a group of finite exponent. Observe that if $K \leqslant_b H \leqslant_b G$ then $K \leqslant_b G$.

Now, the proof of the main theorem.

Proof of Theorem 1. Let G be a periodic $\mathfrak{S}_c$–group, and $\mathcal{N}$ a residual nilpotency system in G such that $(G,\mathcal{N})$ is subnormal-closed. Then G is a Baer group and, by Lemma 1, we may assume that it is a p-group for some prime p. By Theorem 3, G is soluble. We argue by induction on the derived length of G. Thus, by inductive assumption, G' is nilpotent. By Lemma 26 we may also assume that G' is abelian.

Let Θ be the family of all closed subgroups K of G, such that $K \leqslant_b G$. By Lemma 11 there exists $H \in \Theta$, a finite subgroup F of H and a positive integer d, such that all subgroups $F \leqslant S \leqslant H$ with $S \in \Theta$ are subnormal of defect at most d in H.

Let $A = H'$ and, for every $n \in \mathbb{N}$, $A_n = A^{p^n} = \langle x^{p^n} \mid x \in A\rangle$. Then, with the notation of Corollary 25,

$$\bigcap_{n\in\mathbb{N}} A_n = A^{\omega} \leqslant Z(H).$$

Fixed $n \in \mathbb{N}$ and $N \in \mathcal{N}$, write $X = X_{n,N} = A_n(N \cap H)$. Then, all subgroups $S \leqslant_b H$ with $X \leqslant S$ are closed in $(H, \mathcal{N}_H)$ and therefore, by Lemma 6, are closed in G and thus belong to Θ. By choice of H and F it follows that all subgroups $XF \leqslant S \leqslant_b H$ have defect at most d in H. Now, H/X is nilpotent and its derived subgroup, which is contained in AX/X, has finite exponent. It then follows that, if $Y/X = Z(H/X)$, H/Y has finite exponent; in other words, $Y \leqslant_b H$. Moreover, since $Y \geqslant N \cap H$, all subgroups of H/Y containing YF/Y are closed, and so they are subnormal of defect at most d. By Proposition 12, $\gamma_{\beta(d)}(H/Y)$ is finite in H/Y. Then, by a result of P. Hall (see [4, Theorem 4.25]), $\zeta_{2\beta(d)}(H/Y)$ has finite index in H/Y. Since Y/X is the center of H/X, we obtain that $\zeta_{2\beta(d)+1}(H/X)$ has finite index in H/X. This holds for any choice of $n \in \mathbb{N}$ and $N \in \mathcal{N}$.

Now, for any $n \in \mathbb{N}$ and $N \in \mathcal{N}$, let

$$\frac{Z_{n,N}}{A_n(N \cap H)} = \zeta_{2\beta(d)+1}\left(\frac{H}{A_n(N \cap H)}\right).$$

Then $Z_{n,N}$ is closed in H and, by what we have proved above, $Z_{n,N}$ has finite index in H. Thus, letting

$$R = \bigcap_{n \in \mathbb{N},\, N \in \mathcal{N}} Z_{n,N},$$

$(H/R, \mathcal{N}_{H/R})$ is residually finite in the sense of Proposition 14; by the same proposition we deduce that H/R is nilpotent.

Thus, let $c \in \mathbb{N}$ with $\gamma_c(H) \leqslant R$. Then, by definition of R,

$$\gamma_{c+2\beta(d)+1}(H) \leqslant \bigcap_{n \in \mathbb{N},\, N \in \mathcal{N}} A_n(N \cap H).$$

Let $r = c + 2\beta(d) + 1$, and write $W = \gamma_r(H)$. For all $N \in \mathcal{N}$, by applying Corollary 25,

$$\frac{W(N \cap H)}{N \cap H} \leqslant \bigcap_{n \in \mathbb{N}} \left(\frac{A_n(N \cap H)}{N \cap H}\right) = \left(\frac{A(N \cap H)}{N \cap H}\right)^{\omega} \leqslant Z\left(\frac{H}{N \cap H}\right).$$

Hence

$$[W, H] \leqslant \bigcap_{N \in \mathcal{N}} (N \cap H) = 1;$$

thus, $W \leqslant Z(H)$ and, therefore, H is nilpotent.

Finally, let H_G be the largest normal subgroup of G contained in H; then H_G is nilpotent. But, recall that H is subnormal in G and $H \leqslant_b G$; it is therefore not difficult to see that G/H_G has finite exponent. Now, application of Lemma 27 yields that G is nilpotent. □

Theorem 1 is not true for non periodic groups. In fact, as already mentioned in the introduction, H. Smith [3] has shown that there even exist non-nilpotent residually finite (and thus, a fortiori, residually nilpotent) groups in which every subgroup is subnormal.

6. The subnormal intersection property

Let $\mathfrak{S}_\infty$ denote the class of all groups satisfying the subnormal intersection property (*s.i.p.*). Clearly, every finite and every nilpotent group belong to $\mathfrak{S}_\infty$, as well as every group with all subgroups subnormal, or satisfying the minimal condition on subnormal subgroups. Also, it is straigthforward to show that a group in which all subnormal subgroups have defect bounded by a positive integer n (termed a $\mathfrak{B}_n$-group) belongs to $\mathfrak{S}_\infty$. The study of $\mathfrak{S}_\infty$ started with D. Robinson [10], and the proof that a finitely generated soluble group belongs to $\mathfrak{S}_\infty$ if and only if it is finite by nilpotent. Later, Robinson examines wreath products in [11], finding necessary and sufficient conditions on two nilpotent groups H and K, for their (restricted or unrestricted) wreath product $H \wr K$ to belong to $\mathfrak{S}_\infty$. In the same paper, Robinson shows that there exist soluble $\mathfrak{V}_2$-groups with arbitrary derived length, thus making clear that the class of soluble $\mathfrak{S}_\infty$-groups is rather complicated (soluble groups with rank conditions in $\mathfrak{S}_\infty$ are studied in McDougall [12], McCaughan [13] and Franchi [14]).

When moving to locally nilpotent groups, the picture remains in general quite intricate. For example, it has been observed by Leinen [15] that, for every prime p, in the unique countable existentially closed locally finite p-group U_p every subnormal subgroup is normal; so $U_p \in \mathfrak{S}_\infty$. Also, on a less deep level, examples of soluble locally finite p-groups in $\mathfrak{V}_n$ with arbitrary derived length are constructed in [16].

Now, all the p-groups just mentioned are not Baer groups, while Lemma 9 states, in particular, that residually nilpotent $\mathfrak{S}_\infty$-groups are Baer, and our main result (via Corollary 1) shows in turn that, when periodic, they are indeed nilpotent. Thus, the class of Baer groups appears to be a natural following step in studying the s.i.p. In this perspective, we conclude the article by giving a couple of examples that add (in the negative) information on this specific aspect. Both aim at showing that, even for periodic Baer $\mathfrak{S}_\infty$-groups, and in order to force nilpotency, the assumption of residual nilpotency cannot be dropped: we refer, in particular, to Theorem 3 and Proposition 23.

Example 1. A non-soluble Baer p-group satisfying the s.i.p.

Let p be a fixed prime, and C_p a cyclic group of order p. We prove that the P. Hall's generalized wreath product $W = \mathrm{Wr}\, C_p^{\mathbb{N}}$ belongs to the class $\mathfrak{S}_\infty$. We refer to D. Robinson's text [4, §6.2], or to P. Hall's original paper [17], for the background and notations.

In fact, we show that every proper subnormal subgroup of W either contains the derived group W' or is nilpotent; from this it clearly follows that W satisfies the *s.i.p.* (it is easy to check that W is a non-soluble Baer group).

We look at W as the subgroup generated by copies H_n ($n \in \mathbb{N}$) of the cyclic group of order p in a suitable symmetric group, defined (see [4, §6.2]) starting from the regular representation of each H_n and the natural ordering of the set $\mathbb{N}$ of positive integers. For any $n \in \mathbb{N}$, let $I_n = \{0, 1, 2, \dots, n\}$, and $Y_n = \mathbb{N} \setminus I_n$. Then, P. Hall's segmentation lemma says that

$$W = \langle H_0, H_1, \dots, H_n\rangle \wr \langle H_k \mid k > n\rangle = \bigl(\mathrm{Wr}\, C_p^{I_n}\bigr) \wr \bigl(\mathrm{Wr}\, C_p^{Y_n}\bigr) \tag{1}$$

where $\wr$ denotes the standard wreath product. We denote by B_n the base group in the wreath product in the above decomposition (1). Clearly, B_n is nilpotent.

First let N be a normal subgroup of W. We show that either $N \geqslant W'$ or N is contained in B_n, for some $n \in \mathbb{N}$. If $N \nleqslant B_n$ for all n, then, by a lemma in P. Hall's paper (Lemma 6.26 in [4]), $N \geqslant B_n'$ for all $n \in \mathbb{N}$. Since $W = \bigcup_{n\in\mathbb{N}} B_n$, it follows that $N \geqslant W'$.

Now let H be a subnormal subgroup of W, and let $H_o = [W', H]$. Then H_o is normal in W', and so, by another result of P. Hall (Theorem A in [17]), H_o is normal in W. By what we have observed above, either $H_o = W'$ or $H_o \leqslant B_n$ for some n. In the first case, since H is subnormal, we have, for some integer d

$$W' = \bigl[W', H\bigr] = \bigl[W', {}_d H\bigr] \leqslant H;$$

in the second case, H is nilpotent. This clearly implies that W satisfies the *s.i.p.*

Example 2. A non-nilpotent soluble p-group of exponent p^2 satisfying the s.i.p.

Let p be a prime, and let A be a countable elementary abelian p-group, with a fixed base $a_1, a_2, a_3, \dots$

For every $n \geqslant 1$, we define an automorphism x_n of A by setting

$$a_i^{x_n} = \begin{cases} a_i & \text{if } i \equiv 1, \ldots, 2^{n-1} \pmod{2^n}, \\ a_i + a_{i-2^{n-1}} & \text{if } i \equiv 2^{n-1}, \ldots, 2^n \pmod{2^n} \end{cases} \tag{2}$$

and extending by linearity.

It is then easy to check that $x_n^p = 1$ for every n, and, with little more routine work, that $x_n x_m = x_m x_n$, for every $n, m \geqslant 1$. Thus $X = \langle x_n \mid n \geqslant 1 \rangle$ is an infinite elementary abelian p-group of automorphisms of A. For every $k \geqslant 1$ let $A_k = \langle a_1, \ldots, a_k \rangle$ (and, if needed, $A_0 = \{1\}$). We observe that, from definitions (2), the following hold: for every $n \geqslant 1$,

$$[A_{2^n}, x_n] = A_{2^{n-1}} \quad \text{and} \quad [A_{2^n}, x_m] = 1 \quad \text{if } m > n. \tag{3}$$

Let now Y be an infinite subgroup of X. Then, for every $n \geqslant 1$,

$$Y \cap \langle x_{n+1}, x_{n+2}, \ldots \rangle \neq \{1\},$$

and so Y contains an element $y = x_t x_{i_1} \cdots x_{i_r}$, with $t \geqslant n+1$, and $i_j > t$ for every $1 \leqslant j \leqslant r$. From (3) it follows that $x_{i_1} \cdots x_{i_r}$ centralizes A_{2^t}, whence

$$[A, Y] \geqslant [A_{2^t}, y] = [A_{2^t}, x_t] = A_{2^{t-1}} \geqslant A_{2^n}.$$

This holds for every $n \geqslant 1$ and so

$$[A, Y] = A. \tag{4}$$

We now consider the semidirect product $G = A \rtimes X$. Clearly, G is metabelian and has exponent p^2. Also, by (4), $G' = A = [A, G]$ and thus G is not nilpotent. To show that G satisfies the s.i.p., we observe that the following fact holds:

$$\text{If } H \lhd\lhd G, \text{ then either } H \geqslant A = G' \text{ or } AH/A \text{ is finite.} \tag{5}$$

In fact, let H be a subnormal subgroup of G and suppose that AH/A is infinite. Then $AH = AY$, where $Y = AH \cap X$ is an infinite subgroup of X. It then follows from (4) that

$$[A, H] = [A, AH] \geqslant [A, Y] = A.$$

Since $[A, {}_d H] \leqslant H$ for some $d \geqslant 1$, we conclude that $A \leqslant H$, thus proving (5).

Now, having in mind that if $H \lhd\lhd G$ and AH/A is finite then AH is nilpotent, it is clear that G satisfies the s.i.p.

Torsion–free metabelian Baer groups which satisfy s.i.p. and are not nilpotent may also be constructed in a similar way. On the other hand, we have not been able to find an example of a non-soluble Baer group in $\mathfrak{S}_\infty$ of finite exponent, and thus leave open this question. We also mention that it is possible to construct residually – (finite and soluble) groups that are not soluble and satisfy the s.i.p. (in fact, they have the property that every non-trivial subnormal subgroup has finite index and defect at most 4).

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